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ON THE RELATIVE WALL–WITT GROUPS

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ABSTRACT

Let R_* be a simplicial involutive ring. According to certain involutions on $K(R_*)$ and ${}_\varepsilon L(R_*)$, there are $\frac{1}{2}$-local splittings

$$K(R_*) \simeq K^s(R_*) \times K^a(R_*)$$

and

$${}_\varepsilon L(R_*) \simeq {}_\varepsilon L^s(R_*) \times {}_\varepsilon L^a(R_*).$$

It is known [2] that ${}_\varepsilon L^a_n(R_*) \cong {}_\varepsilon L^a_n(\pi_0 R_*) \cong {}_\varepsilon W_n(\pi_0 R_*)$, the Wall–Witt group. Suppose $I \to R \xrightarrow{f} S$ is a split extension of discrete involutive rings with $I^2 = 0$, and I is a free S-bimodule. Then we have

$$K_{n+1}(f) \otimes \mathbb{Q} \cong \mathrm{Prim}_n \bigwedge^* M(I \otimes \mathbb{Q})$$

and

$${}_\varepsilon L_{n+1}(f) \otimes \mathbb{Q} \cong \mathrm{Prim}_n \bigwedge^* {}_\varepsilon\mathcal{O}(I \otimes \mathbb{Q}).$$

The trace map Tr: $\mathrm{Prim}_n \bigwedge^* M(I \otimes \mathbb{Q}) \to \overline{W}_0(\rho_n; I \otimes \mathbb{Q})$ is an isomorphism. We prove in Lemma 1 that the trace map Tr is $\mathbb{Z}/2$-equivariant. In Theorem 2 we show that under a certain assumption the rational relative Wall–Witt group vanishes. Theorem 2 can be extended to a more general case (Theorem 3) by employing Goodwillie's reduction technique [3].

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1. Introduction

T.Goodwillie showed [3] that the rational relative algebraic K-theory is isomorphic to rational relative cyclic homology under a certain assumption: $f\colon R_* \to S_*$ is a map of simplicial rings such that the induced map $\pi_0 R_* \to \pi_0 S_*$ is surjective with nilpotent kernel. He proved it by reducing it to a more special case in which we may explicitly construct an isomorphism Tr: $K_{n+1}(f) \otimes \mathbb{Q} \to W_0(\rho_n; I \otimes \mathbb{Q})$. The map Tr which plays important roles in cyclic homology and classical invariant theory is called a **trace map**. $W_0(\rho_n; I \otimes \mathbb{Q})$ is a certain Hochschild homology. We have

$$K_{n+1}(f) \otimes \mathbb{Q} \cong \mathrm{Prim}_n \bigwedge\nolimits^* M(I \otimes \mathbb{Q}).$$

Assume that all rings that we deal with in this paper are involutive rings (rings with 1 and involutions).

We have involutions on $\mathrm{Prim}_n \bigwedge^* M(I \otimes \mathbb{Q})$ and $\overline{W}_0(\rho_n; I \otimes \mathbb{Q})$ for which the map Tr is $\mathbb{Z}/2$-invariant, i.e.,

$$\mathrm{Tr}\colon K^s_{n+1}(f) \otimes \mathbb{Q} \overset{\cong}{\to} \overline{W}{}^+_0(\rho_n; I \otimes \mathbb{Q}).$$

Meanwhile, the author proved in [9] that under some condition the rational relative hermitian K-theory is isomorphic to rational relative dihedral homology. In the proof it is shown that Tr: ${}_\varepsilon L_{n+1}(f) \otimes \mathbb{Q} \to \overline{W}{}^+_0(\rho_n; I \otimes \mathbb{Q})$ is an isomorphism. According to an involution on ${}_\varepsilon L_{n+1}(f) \otimes \mathbb{Q}$ we have

$${}_\varepsilon L_{n+1}(f) \otimes \mathbb{Q} \cong {}_\varepsilon L^s_{n+1}(f) \otimes \mathbf{Q} \oplus {}_\varepsilon L^a_{n+1}(f) \otimes \mathbb{Q}.$$

Burghelea and Fiedorowicz [2] proved that

$${}_\varepsilon L^s_{n+1}(f) \otimes \mathbb{Q} \cong K^s_{n+1}(f) \otimes \mathbb{Q},$$

via the isomorphism induced by the forgetful map. ${}_\varepsilon L^a_{n+1}(f) \otimes \mathbb{Q}$ is isomorphic to the relative Wall–Witt theory ${}_\varepsilon W_{n+1}(f) \otimes \mathbb{Q}$, which is another face of the obstruction group of Wall in surgery theory. Considering all these facts, we can prove that the rational relative Wall–Witt theory vanishes.

2. Preliminaries

First recall a bimodule structure (cf. [3]). Let R be a ring with 1. Let M be an involutive R-bimodule (R-bimodule with an involution). We can endow $M^{\otimes n}$

with an $R^{\otimes n}$-bimodule structure according to a permutation $\rho \in \Sigma_n$ as follows:

$$\begin{aligned}(r_1 \otimes \cdots \otimes r_n)(a_1 \otimes \cdots \otimes a_n) &= r_1a_1 \otimes \cdots \otimes r_na_n, \quad r_i \in R,\ a_i \in M,\\ (a_1 \otimes \cdots \otimes a_n)(r_1 \otimes \cdots \otimes r_n) &= a_1r_{\rho(1)} \otimes \cdots \otimes a_nr_{\rho(n)}.\end{aligned}$$

Define $D^n(\rho; M)$ to be $M^{\otimes n}$ with this $R^{\otimes n}$-bimodule structure. There is an involution $\imath\colon D^n(\rho; M) \to D^n(\rho; M)$ defined by

$$\imath\colon a_1 \otimes \cdots \otimes a_n \mapsto (-1)^{n(n+1)/2}\bar{a}_n \otimes \bar{a}_{n-1} \otimes \cdots \otimes \bar{a}_1.$$

Denote the Hochschild homology $H_*(R^{\otimes n}; D^n(\rho; M))$ by $W_*(\rho; M)$.

Note that $W_0(\rho; M)$ is the quotient of $M^{\otimes n}$ by the relations

$$r_1a_1 \otimes \cdots \otimes r_na_n \sim a_1r_{\rho(1)} \otimes \cdots \otimes a_nr_{\rho(n)}.$$

Denote the class of $a_1 \otimes \cdots \otimes a_n$ in $W_0(\rho; M)$ by $[a_1, \ldots, a_n]_\rho$. For each permutation $\lambda \in \Sigma_n$ we have an isomorphism $W_0(\rho; M) \to W_0(\lambda^{-1}\rho\lambda; M)$ defined by $[a_1, \ldots, a_n]_\rho \mapsto \mathrm{sgn}(\lambda)[a_{\lambda(1)}, \ldots, a_{\lambda(n)}]_{\lambda^{-1}\rho\lambda}$. Thus the centralizer $C(\rho)$ of ρ acts on $W_0(\rho; M)$. $\overline{W}_0(\rho; M)$ denotes the covariant of $C(\rho)$-action, and $\overline{[a_1, \ldots, a_n]_\rho} \in \overline{W}_0(\rho; M)$ means the class of $[a_1, \ldots, a_n]_\rho$. We may have an involution on $\overline{W}_0(\rho; M)$ induced by that on $D^n(\rho; M)$. Denote by $\overline{W}_0^+(\rho; M)$ the $\mathbb{Z}/2$-invariant of $\overline{W}_0(\rho; M)$.

Let R_* be a simplicial involutive ring. Recall that $K_i(R_*) = \pi_i B\widehat{GL}(R_*)^+$ for $i \geq 1$, where $\widehat{GL}(R_*)$, which is a grouplike simplicial monoid, is a collection of connected components of all matrices $M(R_*)$ so that $\pi_0\widehat{GL}(R_*) = GL(\pi_0 R_*)$. We denote $B\widehat{GL}(R_*)^+$ by $K(R_*)$. We define the hermitian K-theory ${}_\varepsilon L_i(R)$ as follows (cf. [2]): Let $\varepsilon = \pm 1$. Let ${}_\varepsilon\widehat{O}_{n,n}(R_*)$ be a simplicial category whose objects are $2n \times 2n$ matrices A over R_* such that $A = \varepsilon A^*$ and $p(A) = C^* \begin{pmatrix} 0 & \varepsilon I \\ I & 0 \end{pmatrix} C$ for some $C \in GL_{2n}(\pi_0 R)$, where the map p is induced by the quotient map $R_* \to \pi_0 R_*$. A morphism $M\colon A \to B$ is a matrix $M \in \widehat{GL}_{2n}(R_*)$ with $M^*AM = B$. Let ${}_\varepsilon\widehat{O}(R_*)$ be $\coprod_{n\geq 1} {}_\varepsilon\widehat{O}_{n,n}(R_*)$. By $\widetilde{\mathrm{Sym}}^{\varepsilon}_{n,n}(R_*)$ we denote the set of objects of the category ${}_\varepsilon\widehat{O}_{n,n}(R_*)$. Note that ${}_\varepsilon\widehat{O}(R_*)$ is a permutative category via the direct sum of matrices, so the group completion of the classifying space $B_\varepsilon\widehat{O}(R_*) = \coprod_{n\geq 1} B\,{}_\varepsilon\widehat{O}_{n,n}(R)$ is an infinite loop space which is the plus construction. Denote

$${}_\varepsilon L(R_*) = B_\varepsilon\widehat{O}(R_*)^+ = \Omega B_\oplus\Big(\coprod_{n\geq 1} B\,{}_\varepsilon\widehat{O}_{n,n}(R_*)\Big).$$

Define the i-th hermitian K-theory ${}_\varepsilon L_i(R_*)$ to be

$$\pi_i\,{}_\varepsilon L(R_*) = \pi_i B\,{}_\varepsilon\widehat{O}(R_*)^+.$$

Meanwhile, $B\,{}_\varepsilon\widehat{O}_{n,n}(R_*)$ is identified with the two-sided bar construction

$$B(\widetilde{\mathrm{Sym}}^{\varepsilon}_{n,n}(R_*), \widehat{GL}_{2n}(R_*), *).$$

$\widehat{GL}_{2n}(R_*)$ acts on $\widetilde{\mathrm{Sym}}^{\varepsilon}_{n,n}(R_*)$ via $(A, M) \mapsto M^*AM$ where $M \in \widehat{GL}_{2n}(R_*)$, $A \in \widetilde{\mathrm{Sym}}^{\varepsilon}_{n,n}(R_*)$. We denote $B{}_\varepsilon\widehat{O}(R_*)^+$ by ${}_\varepsilon L(R_*)$.

Let R be an (discrete) involutive ring. We have an involution on $K(R) = BGL(R)^+$ induced by the map $A \mapsto (A^*)^{-1}$ on $GL_n(R)$. More generally, for a simplicial involutive ring R_*, the involution on $K(R_*)$ is induced by the involution on $B\widehat{GL}_n(R_*)$ sending

$$[(A_1, A_2, \ldots, A_k), (t_0, \ldots, t_k)]$$

to

$$[(A_k^*, A_{k-1}^*, \ldots, A_1^*), (t_k, t_{k-1}, \ldots, t_0)].$$

Recall that for a group G the involution on BG induced by the inverse map on G is canonically homotopic to the map given by

$$[(g_1, g_2, \ldots, g_k), (t_0, \ldots, t_k)] \mapsto [(g_k, g_{k-1}, \ldots, g_1), (t_k, t_{k-1}, \ldots, t_0)].$$

The involution on $B\,{}_\varepsilon\widehat{O}_{n,n}(R_*) = B(\widetilde{\mathrm{Sym}}^{\varepsilon}_{n,n}(R_*), \widehat{GL}_{2n}(R_*), *)$ is defined by the involution on $\widetilde{\mathrm{Sym}}^{\varepsilon}_{n,n}(R_*)$ sending A to $-A$. We have $\frac{1}{2}$-local splittings:

$$K(R_*) \simeq K^s(R_*) \times K^a(R_*) \quad \text{and} \quad {}_\varepsilon L(R_*) \simeq {}_\varepsilon L^s(R_*) \times {}_\varepsilon L^a(R_*),$$

where s and a denote *the symmetric* and *the antisymmetric part*, respectively. The forgetful map

$$B(\widetilde{\mathrm{Sym}}^{\varepsilon}_{n,n}(R_*), \widehat{GL}_{2n}(R_*), *) \to \widehat{GL}_{2n}(R_*)$$

induces the map ${}_\varepsilon L(R_*) \to K(R_*)$ which, by being restricted on ${}_\varepsilon L^s(R_*)$, makes a homotopy equivalence ${}_\varepsilon L^s(R_*) \to K^s(R_*)$ [2]. It is also shown in [2] that ${}_\varepsilon L^a(R_*) \simeq {}_\varepsilon L^a(\pi_0 R_*)$, while $\pi_n\,{}_\varepsilon L^a(\pi_0 R_*)$ is isomorphic to the Wall–Witt group ${}_\varepsilon W_n(\pi_0 R_*)$, that is, the Wall–Witt group of a simplicial ring R_* means that of $\pi_0 R_*$ which is a *discrete* ring.

Let R be an involutive ring with 1. For a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{2n}(R)$, denote $\begin{pmatrix} \delta^* & \varepsilon\beta^* \\ \bar{\varepsilon}\gamma^* & \alpha^* \end{pmatrix}$ by $A^\# = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\#$. Recall that the hermitian orthogonal group ${}_\varepsilon O_{n,n}(R)$ is defined by

$$ {}_\varepsilon O_{n,n}(R) = \left\{ M \in M_{2n}(R) \colon MM^\# = M^\# M = 1 \right\}. $$

The Lie algebra corresponding to ${}_\varepsilon O_{n,n}(R)$ is defined and denoted by

$$ \begin{aligned} {}_\varepsilon \mathcal{O}_{n,n}(R) &= \{ M \in M_{2n}(R) \colon -M^\# = M \} \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{pmatrix} \in M_{2n}(R) \;\middle|\; \beta = -\varepsilon\beta^*, \ \gamma = -\bar{\varepsilon}\gamma^* \right\} \\ &= \left\{ M \in M_{2n}(R) \colon M^* \begin{pmatrix} 0 & \varepsilon I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon I \\ I & 0 \end{pmatrix} M^* = 0 \right\}; \end{aligned} $$

${}_\varepsilon \mathcal{O}(R)$ is defined by $\varinjlim {}_\varepsilon \mathcal{O}_{n,n}(R)$.

3. Main Theorem

Suppose that $I \to R \xrightarrow{f} S$ is a split extension of discrete involutive rings with $I^2 = 0$, and I is a free S-bimodule. Then

$$ 0 \to M(I) \to GL(R) \to GL(S) \to 1 $$

and

$$ 0 \to {}_\varepsilon \mathcal{O}(I) \to {}_\varepsilon O(R) \to {}_\varepsilon O(S) \to 1 $$

are exact sequences. The rational relative algebraic K-theory $K_{n+1}(f) \otimes \mathbb{Q}$ is isomorphic to $\pi_n BM(I) \otimes \mathbb{Q}$ [3], so we have

$$ K_{n+1}(f) \otimes \mathbb{Q} \cong \mathrm{Prim}_n \wedge^* M(I \otimes \mathbb{Q}). $$

Similarly, the rational relative hermitian K-theory ${}_\varepsilon L_{n+1}(f) \otimes \mathbb{Q}$ is isomorphic to $\mathrm{Prim}_n \bigwedge_\varepsilon^* \mathcal{O}(I \otimes \mathbb{Q})$ ([9]). Goodwillie showed [3] that

$$ K_{n+1}(f) \otimes \mathbb{Q} = \mathrm{Prim}_n \wedge^* M(I \otimes \mathbb{Q}) $$

is isomorphic to $\overline{W}_0(\rho_n; I \otimes \mathbb{Q})$ via the trace map

$$ \mathrm{Tr} \colon \mathrm{Prim}_n \wedge^* M(I \otimes \mathbb{Q}) \to \overline{W}_0(\rho_n; I \otimes \mathbb{Q}) $$

that is given by

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n \mapsto \sum_{\{i_j\}} \overline{[A_1(i_1,i_2),\cdots,A_n(i_n,i_1)]_{\rho_n}}.$$

There is an involution τ on $K_{n+1}(f)\otimes\mathbb{Q} = \mathrm{Prim}_n \wedge^* M(I\otimes\mathbf{Q})$ which is induced by $A \mapsto -A^*$ for $A \in M(I\otimes\mathbb{Q})$. $\overline{W}_0(\rho_n; I\otimes\mathbb{Q})$ also has an involution σ given by

$$\sigma\colon \overline{[a_1,\ldots,a_n]_{\rho_n}} \mapsto (-1)^{n(n+1)/2}\overline{[\bar{a}_n,\ldots,\bar{a}_1]_{\rho_n}}.$$

LEMMA 1: *The trace map* Tr: $\mathrm{Prim}_n \bigwedge^* M(I\otimes\mathbb{Q}) \to \overline{W}_0(\rho_n; I\otimes\mathbb{Q})$ *is* $\mathbb{Z}/2$-*equivariant.*

Proof: We have the following commutative diagram.

$$\begin{array}{ccc}
A_1 \wedge A_2 \wedge \cdots \wedge A_n & \longrightarrow & \sum_{\{i_j\}} \overline{[A_1(i_1,i_2),\ldots,A_n(i_n,i_1)]_{\rho_n}} \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} \\
(-1)^n A_1^* \wedge A_2^* \wedge \cdots \wedge A_n^* & \longrightarrow & (-1)^n \sum_{\{i_j\}} \overline{[A_1^*(i_1,i_2),\ldots,A_n^*(i_n,i_1)]_{\rho_n}} \\
 & & \| \\
 & & = (-1)^n(-1)^{\frac{n(n+1)}{2}} \sum_{\{i_j\}} \overline{[\overline{A_n(i_n,i_1)},\ldots,\overline{A_1(i_1,i_2)}]_{\rho_n}}.
\end{array}$$

∎

Lemma 1 implies there is an isomorphism

$$K^s_{n+1}(f)\otimes\mathbb{Q} \xrightarrow[\cong]{\mathrm{Tr}} \overline{W}_0^+(\rho_n; I\otimes\mathbb{Q}). \tag{1}$$

This enables us to obtain an important result on higher Wall–Witt groups.

THEOREM 2: *Suppose that* $I \to R \xrightarrow{f} S$ *is a split extension of discrete involutive rings with* $I^2 = 0$, *and* I *is a free* S-*bimodule. Then the rational relative Wall-Witt group vanishes, i.e.,* ${}_\varepsilon W_n(f)\otimes\mathbb{Q} = 0$ *for* $n \geq 1$.

Proof: The trace map Tr induces an isomorphism (cf. [9])

$${}_\varepsilon L_{n+1}(f)\otimes\mathbb{Q} \xrightarrow[\cong]{\mathrm{Tr}} \overline{W}_0^+(\rho_n; I\otimes\mathbb{Q}). \tag{2}$$

Burghelea and Fiedorowicz [2] showed that ${}_\varepsilon L^s_{n+1}(f)\otimes\mathbb{Q}$ is isomorphic to $K^s_{n+1}(f)\otimes\mathbb{Q}$ via the isomorphism induced by the forgetful map. The antisymmetric part ${}_\varepsilon L^a_{n+1}(f)\otimes\mathbb{Q}$ is isomorphic to ${}_\varepsilon W_{n+1}(f)$. Considering the commutative diagram that consists of the trace maps $((1),(2))$, and the forgetful map, we get ${}_{\varepsilon}W_n(f)\otimes\mathbb{Q}=0$ for $n\geq 1$. ∎

Let F: ${}_{\varepsilon}L_n(f)\otimes\mathbb{Q}\to K_n(f)\otimes\mathbb{Q}$ be the map induced by the forgetful map ${}_{\varepsilon}L(f)\to K(f)$. Define $\Phi(f)$: ${}_{\varepsilon}L_n(f)\otimes\mathbb{Q}\to K_n^s(f)\otimes\mathbb{Q}$ by the composite

$$ {}_{\varepsilon}L_n(f)\otimes\mathbb{Q}\xrightarrow{F}K_n(f)\otimes\mathbb{Q}\xrightarrow{p_1}K_n^s(f)\otimes\mathbb{Q} $$

where p_1 is the projection. Theorem 2 implies that $\Phi(f)$ is an isomorphism. We can get a more general result than Theorem 2 by employing the technique analogous to Goodwillie's. We get the following theorem using the naturality of the map $\Phi(f)$.

THEOREM 3: *Suppose f: $R\to S$ is a map of simplicial involutive rings such that the induced map $\pi_0R\to\pi_0S$ is surjective with nilpotent kernel. Then the rational relative Wall–Witt group vanishes, i.e., ${}_{\varepsilon}W_n(f)\otimes\mathbb{Q}=0$ for $n\geq 1$.*

Proof: It suffices to show that $\Phi(f)$ is an isomorphism, since the following diagram commutes (cf. [2], Theorem 4.3):

$$ \begin{array}{ccccc} {}_{\varepsilon}L_n(f)\otimes\mathbb{Q} & \xrightarrow{F} & K_n(f)\otimes\mathbb{Q} & \xrightarrow{p_1} & K_n^s(f)\otimes\mathbb{Q} \\ & {\scriptstyle q_1}\searrow & & \nearrow{\scriptstyle F} & \\ & & {}_{\varepsilon}L_n^s(f)\otimes\mathbb{Q} & & \end{array} $$

Here q_1 is the projection. We show why Theorem 2 implies Theorem 3.

STEP I: We first show that it suffices to prove Theorem 3 for a surjection f: $R\longrightarrow S$ with square-zero kernel. For a commutative diagram of simplicial rings satisfying the assumption of Theorem 3

$$ \begin{array}{ccc} R & \xrightarrow{f} & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & T & \end{array} $$

if Theorem 3 holds for any two of these three maps then it holds for the other one. (From the commutative diagram

$$\begin{array}{ccc} K(R) & \longrightarrow & K(S) \\ \downarrow & & \downarrow \\ K(T) & = & K(T) \end{array}$$

we get a long exact sequence

$$\cdots \to K_n(f) \to K_n(h) \to K_n(g) \to K_{n-1}(f) \to \cdots$$

Likewise we get the same long exact sequence of relative hermitian K-theory, so we can use the five lemma and the surjectivity of $K_1(R) \to K_1(S)$.) Now we apply this principle to the diagram

$$\begin{array}{ccc} R & \longrightarrow & R/I^n \\ & \searrow \quad \swarrow & \\ & R/I^{n-1} & \end{array}$$

Using induction we see that if Theorem 3 holds for surjection $R \longrightarrow R/I$ with $I^2 = 0$, then it also holds for surjections with $I^n = 0$ for $n \geq 2$.

Now consider the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ \pi_0 R & \longrightarrow & \pi_0 S \end{array}$$

Then by the above principle again it suffices to consider the three maps of the above diagam other than f. The theorem holds for the vertical maps which are the quotient maps (see [3], 375-376).

Step II: The relative K-theory and relative hermitian K-theory with *square-zero* simplicial ideal can be computed dimensionwise ([3], Lemma I.2.2 and [9], Lemma 3.6) so it is enough to consider $\Phi(f)$ for the extension $I \to R \xrightarrow{f} S$ of *discrete* rings with $I^2 = 0$. Moreover, we may assume that S is free, since we can replace S by its free resolution. Thus we may assume that the extension is split. For any ideal I of R, if $I^2 = 0$ then I is an R/I-bimodule. We now complete the proof of this theorem by replacing I by its free resolution. ∎

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